

Transitions To the Long-Resident State in coupled chaotic oscillators

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The behaviors of coupled chaotic oscillators before complete synchronization were investigated. We report three phenomena: (1) The emergence of long-time residence of trajectories besides one of the saddle foci; (2) The tendency that orbits of the two oscillators get close becomes faster with increasing the coupling strength; (3) The diffusion of two oscillator's phase difference is first enhanced and then suppressed. There are exact correspondences among these phenomena. The mechanism of these correspondences is explored. These phenomena uncover the route to synchronization of coupled chaotic oscillators.

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Synchronization is a universal and fundamental behavior occurring in various fields[1, 2, 3]. Different synchronous states can be observed in coupled chaotic systems such as complete synchronization (CS)[4, 5], generalized synchronization (GS)[7], phase synchronization (PS)[8, 9], measure synchronization (MS)[6] and so on. What people care about in studies of synchronization is often the critical coupling, in which the synchronization can achieve[10], and the behavior near this threshold. The dynamic in regimes far from the global synchronous state is related to the mechanism that makes the synchronization be a stable state. In this paper we report three phenomena which can help us to understand the problem above. We also uncover a mechanism of synchronization in systems which have two or more saddle foci.

When two or more chaotic systems were linearly coupled into a network, their differential equation can be represented by $\dot{\vec{X}} = \vec{F}(\vec{X}) + \varepsilon \Gamma \otimes C \vec{X}$, where $\vec{X} = (\vec{x}^1, \vec{x}^2, \dots, \vec{x}^N)$, and N is the node number of network. ε denotes the coupling strength. $\Gamma : \mathbb{R}^m \rightarrow \mathbb{R}^m$ characterizes the coupling schemes among the variables of the nodes in a network. $C = M - D$, where M denotes the adjacency matrix of the network (the element M_{ij} denotes the number of the edges that link node i and j), D is a diagonal matrix, and satisfies $D_{ii} = \sum_{j=1}^N M_{ij}$. In this paper, we take the N Lorenz systems ($\dot{x} = \sigma(y - x), \dot{y} = rx - y - xz, \dot{z} = xy - \beta z$, where $\sigma = 10, r = 28, \beta = 8/3$) as our nodes and couple them into an array (the periodic boundary condition is applied) with $\Gamma_{ij} = 0$ ($i, j = 1, 2, 3$) except $\Gamma_{11} = 1$.

There are three fixed points in phase space of a single Lorenz oscillator. Two of them are saddle foci ($\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, r-1$) and another is a saddle node (0,0,0). The trajectory usually revolves around the two saddle foci alternatively. Fig.1(a) is the time serial of x -component of a single Lorenz system. The revolution about one of the saddle foci can be regarded as one

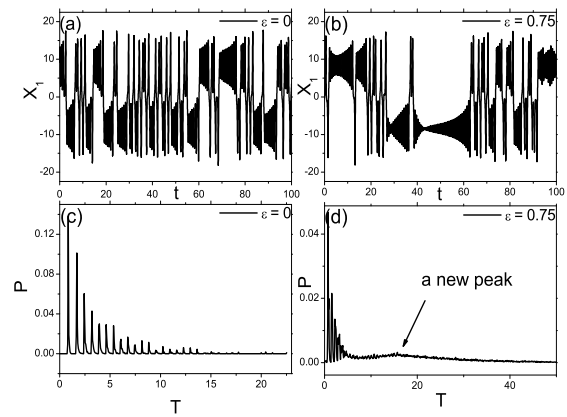


FIG. 1: (a):Time series $x_i(t)$ for a single Lorenz system. (b):Residence time statistics $P(T)$. (c)(d):Coupled Lorenz system($N = 2$), $\varepsilon = 0.75$.

dynamic phase. Fig.1(b) is the statistics $P(T)$ of the residence time T that the system resides in a phase. This distribution consists of several almost discrete peaks. When the two systems are coupled together, new phenomenon can be observed. Fig.1(c) is one of the time series of x -component in a system consisting of two Lorenz oscillators. It can be observed that the residence time in certain phase is longer than the situation in Fig.1(a). Accordingly, a new wide continuous smooth peak appears in the distribution of residence time in Fig.1(d). The above phenomenon suggests a new state that we call long-residence state (LRS) in coupled chaotic systems. While the new peak in the distribution indicate the emergence of LRS, one can define $P_0 = \int_{T_0}^{+\infty} P(T) dT$ as the probability of resident time that is larger than T_0 . For a larger T_0 , P_0 is a measure of the emergence of LRS. Fig.2(a) is the relationship between ε and P_0 ($T_0 = 10$). Obviously, for a weak coupling, $P_0 \approx 0$; But when the coupling ε is greater than a critical point $\varepsilon_1 \approx 0.36$, P_0 increase sharply, suggesting the emergence of the LRS.

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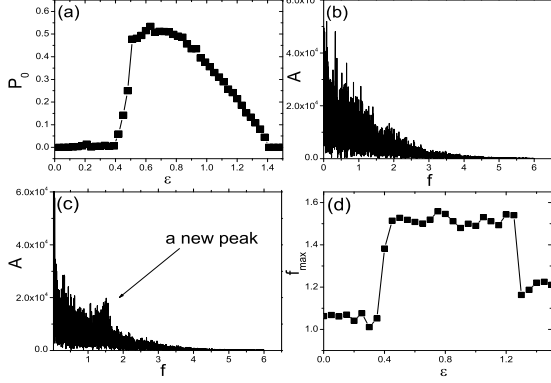


FIG. 2: (a): $P_0 \sim \varepsilon$, $T_0 = 10$. (b): Power spectrum $A(f)$ of $x(t)$ for a single Lorenz system. (c): $A(f)$ of $x_1(t)$ for coupled Lorenz system ($N = 2$), $\varepsilon = 0.75$. (d): $f_{max} \sim \varepsilon$, $f_0 = 1$.

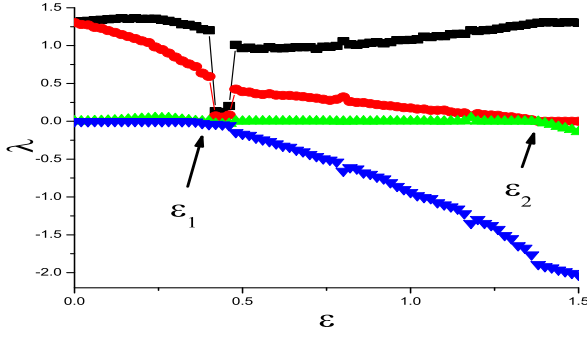


FIG. 3: The four largest LEs for $N = 2$ Lorenz oscillators. There are two critical coupling strength $\varepsilon_1 = 0.36$ and $\varepsilon_2 = 1.38$.

On the other hand, the emergence of LRS can also be observed through the power spectrum of the chaotic motion. Fig.2(b) is the spectrum amplitude of a single Lorenz system, and Fig.2(c) gives that of a coupled system with two oscillators. A new peak appears in the high-frequency regime. The relationship between this frequency peak in the region $[f_0, \infty)$ and coupling strength ε is plotted in Fig.2(d). A critical coupling strength ε_1 can also be found, indicating the emergence of LRS, and the critical point is same as that in Fig.2(a).

To uncover what happens near the critical point of LRS, it is instructive to study the Lyapunov exponent(LE) spectrum is useful. Fig.3 is the four largest LEs of two coupled Lorenz oscillators. At $\varepsilon_1=0.36$ and $\varepsilon_2=1.38$, one of the zero exponents is found to become negative respectively. Therefore, the LRS is accompanied by a topological transition of the chaotic attractor and is an intrinsic bifurcation embedded in complicated motion. One of the Lyapunov exponents becoming negative means the decrease of the dimension

of the chaotic attractor. On the other hand, when CS achieve at ε_2 , the state of the coupled system is on an invariant synchronous sub-manifold, and the dimension of the attractor will become one half of that for uncoupled system. Are there any relations between LRS and the change of the Lyapunov exponents. To answer the question, let's study the evolution of the trajectory distance between two oscillators. We define the Euclidean distance as $r(t) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. When the synchronization is achieved, $r(t)$ will tend to 0 rapidly. In the non-synchronous range, the behavior of $r(t)$ is complex, and we prefer to studying its statistical characters. By defining $p(r)dr$ as the probability of the distance r located in $r \rightarrow r + dr$, the accumulative distribution $P(R) = \int_0^R p(r)dr$ describes the portion of trajectory distance that is smaller than R . In the non-synchronous regime $P(R) \rightarrow 0$ for $R \rightarrow 0$, and $P(R) \rightarrow 1$ when $R \rightarrow R_{max}$, where R_{max} is the determined by the attractor size. When CS is achieved, one has $P(R) = 1$. Fig.4(a) shows the behavior of $P(R)$ for different coupling strengths. It is interesting that $P(R)$ obeys a power law for $R \ll 1$, i.e. $P(R) \propto R^\alpha$. This indicates $p(r) \propto r^{\alpha-1}$, implying $p(r)$ may change from a Gaussian-like function to a Poisson-like function when α decreases below 1 ($p(r) \rightarrow 0$ when $r \rightarrow R_{max}$). The variation of the exponent versus the coupling ε is shown in Fig.4(b). It is very interesting to notice that at $\varepsilon < \varepsilon_1$, $\alpha > 1$, and when $\varepsilon > \varepsilon_1$, $\alpha < 1$. This manifests the emergence of the long-resident state.

the emergence of LRS implies longer rotations before switching to another rotating saddle focus. This should be closely related to the phase dynamics of coupled Lorenz oscillators. In Lorenz system, a dynamical phase $\theta_i = \text{tg}^{-1}[u_y^i(t)/u_x^i(t)]$ can be introduced, where $u_x^i(t) = z_i - (r - 1)$ and $u_y^i(t) = \sqrt{x_i^2 + y_i^2} - \sqrt{2\beta(r - 1)}$, ($i = 1, 2, \dots, N$). The definition of PS is that the average frequency of two or more oscillator is equal to each other, so the synchronization is in terms of the "average" of the phase. In fact, the phase defined above is not a well-defined quantity (no better definition of a single phase been proposed for oscillations with multiple rotation centers). Therefore diffusion process can be observed for the phase difference between two oscillators, i.e. $\Delta\theta = \theta_i - \theta_j$, $i, j = 1, 2, \dots, N$ are time-dependent. To understand the changes of diffusive process, we calculate the second center moment of the difference $\langle \Delta\theta^2 \rangle - \langle \Delta\theta \rangle^2$, where $\langle \cdot \rangle$ means the average of ensembles. Fig.4(c) describes the evolution of the difference at $\varepsilon = 0.75$. It is approximately proportional to time, therefore this process is the Brownian motion. We can further define the diffusion coefficient,

$$D = \lim_{T \rightarrow \infty} \frac{1}{2T} [\langle \Delta\theta^2 \rangle - \langle \Delta\theta \rangle^2]$$

In Fig.4(d), we give the behavior of the diffusion coefficient D against the coupling ε . It can be found that the diffusion is first enhanced for weak couplings and then depressed for larger couplings. There is a peak near the

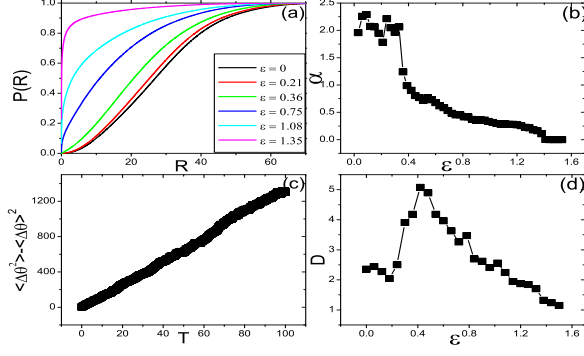


FIG. 4: $N = 2$ (a): $P(R) \sim R$. (b): $\alpha \sim \varepsilon$. (c): The behavior of $\langle \Delta\theta^2 \rangle - \langle \Delta\theta \rangle^2$. (d) The diffusion coefficient D against the coupling strength ε .

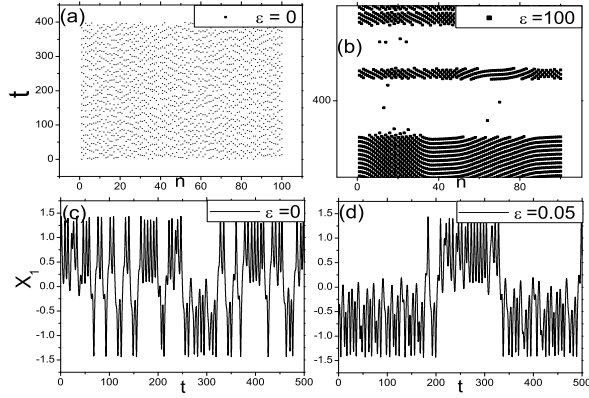


FIG. 5: (a)(b): The spatiotemporal behavior of a lattice of $N = 100$, $\varepsilon = 0$ [(a)], $\varepsilon = 100$ [(b)]. (c)(d): The evolution of $x(t)$ for two coupled Duffing oscillators ($\ddot{x} + 0.5\dot{x} + x^3 - x - 0.4\cos(t) = 0$) with different ε .

coupling ε_1 where the LRS emerges. This correspondence implies that the appearance of LRS does favor to PS. Because there are two rotating centers in a single Lorenz system, a small coupling will do harm to the synchronization of coupled system[12]. When the system is in the LRS, oscillators usually prefer to staying in one of the phase, and the trajectories of two oscillators are easy to close.

In fact, a system with more oscillators may exhibits stronger effect of LRS. Figs.5(a),(b) gives the spatiotemporal patterns at $\varepsilon = 0$ and $\varepsilon = 100$ respectively, for $N = 100$. The oscillator is labeled when it is in one scroll. One can find that the spatiotemporal pattern at $\varepsilon = 0$ is a random pattern, but obviously a very long stay of oscillators in one scroll can be found for $\varepsilon = 100$.

Finally, we want to stress that the LRS we observed here is generic. It can be observed in different situations. We have checked coupled Duffing oscillators(as shown in Figs.5(c) and (d)), the residence time of two coupled Duffing oscillators is obviously longer than that of a single oscillator), Chua circuits, and Chen's oscillators [13]. All these systems exhibit the LRS behaviors for moderate couplings. Even if there are some parameter mismatches, a LRS can still be observed. Furthermore, we also find that the LRS does not depend on the coupling forms, e.g., x -, y - or z -coupling, and local or global.

In conclusion, we explored the dynamics of coupled chaotic oscillators with multiple scrolls(saddle foci) prior to global synchronization. We find a generic transition to the long-resident state (LRS), where the oscillations experience a rather long duration in one scroll. This transition is manifested by several different behaviors. First, one zero LE in LE spectrum become negative. Second, the transition to LRS is accompanied by the qualitative changes in the trajectory-distance distribution and power spectra. Third, we show that the emergence of the LRS is accompanied by the enhancement-depression transition of the diffusion of the phase difference. The phenomenon of LRS is generic, i.e., it is irrelevant to the dynamics, number, and the interaction types of oscillators.

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